

# Bulk Witten Indices from $D = 10$ Yang Mills Integrals

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## Abstract

Values for the bulk Witten indices for  $D = 10$  Yang-Mills integrals for regular simple groups of rank 4 and 5 are calculated by employing the BRST deformation technique by Moore, Nekrasov and Shatashvili. The results cannot be reconciled with the double assumption that the number of normalizable ground states is given by certain simple partition functions given by Kac and Smilga as well as that the corresponding boundary term is always negative.

## 1 Introduction

Supersymmetric Yang-Mills theories in  $D$  dimensions are described by an action of the form

$$S = \int d^D x \left( -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a + \frac{i}{2} \bar{\psi}_\alpha^a \gamma_{\alpha\beta}^\mu D_\mu^{ab} \psi_\beta^b \right) \quad (1)$$

where indices  $a, b, \dots$  correspond to the adjoint representation of some semisimple compact Lie algebra,  $\mu, \nu, \dots$  are spacetime vector indices (spatial indices will be denoted by  $i, j, \dots$ ), and  $\alpha, \beta, \dots$  are spinorial spacetime indices; furthermore, the nonabelian field strength is formed from the vector potential  $A_\mu^a$  via

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{bc}^a A_\mu^b A_\nu^c, \quad (2)$$

$f_{bc}^a$  being the gauge group structure constants, and the gauge covariant derivative is given by

$$D_\mu^{ab} \psi_\beta^b = \delta^{ab} \partial_\mu \psi_\beta^b + g f_{cd}^b A_\mu^c \psi_\beta^d. \quad (3)$$

The equality of the number of physical bosonic and fermionic degrees of freedom required by supersymmetry furthermore forces the number of transversal modes of the gauge field  $A^a$ ,  $D - 2$ , to be a power of two, as is the size of the corresponding irreducible spinor representation of the Lorentz algebra. One finds that it is in fact possible here to implement supersymmetry in  $D = 3, 4, 6, 10$ . By truncation to configurations that have no spatial dependency (i.e. dimensional

reduction to zero space dimensions, see e.g. [6, 14]), one obtains supersymmetric quantum mechanics [3, 5, 10, 24] with a Hamiltonian

$$H = \frac{1}{2} P_i^a P_i^a + \frac{g^2}{4} f_{bca} f_{dea} A_i^b A_j^c A_i^d A_j^e + \frac{ig}{2} f_{abc} \psi_\alpha^a \Gamma_{\alpha\beta}^i \psi_\beta^b A_i^c \quad (4)$$

with  $P_i^a = -i \frac{\partial}{\partial A_i^a}$ .

The  $D = 10$  case with gauge group  $SU(N)$  regained a lot of popularity after initial work by de Wit, Hoppe, and Nicolai [7] who showed the relevance of this particular model for the description of the eleven-dimensional supermembrane (where  $SU(N)$  appears as a regularized version of the Lie group of area-preserving membrane diffeomorphisms) through the so-called M(atrix) Theory Conjecture [4], which states that this Hamiltonian of a system of  $N$   $D0$ -branes of type IIA string theory should give a complete, non-perturbative description of the dynamics of  $M$ -theory in the light cone frame.<sup>1</sup> Furthermore, these matrix models play an important role in the IKKT model [13], which may provide a non-perturbative description of IIB superstring theory.

Gauge groups of other types are also of interest here, as realizations of this model with  $SO(N)$  and  $Sp(2N)$  symmetry are given by systems of  $N$  type-IIA  $D0$ -branes moving in orientifold backgrounds, cf. [12].

One question of chief importance is that of normalizable zero energy vacuum states of these models; this is difficult to settle as the potentials have flat valleys that extend to infinity, and hence, the corresponding ground states are at threshold. While it is exceedingly difficult to try to explicitly solve the Schrödinger equation for these models, it is already of great interest to know the *number* of such normalizable ground states; for example, it is of crucial importance to the Matrix Theory conjecture that there is exactly one such state for every  $N$  in the models derived from  $SU(N)$  gauged  $\mathcal{N} = 1$  SYM (which just corresponds to a bound state of  $N$   $D0$ -branes that appears as a graviton with  $N$  units of momentum in the compactified direction). While this is widely believed to be the case by now, the situation is still much less clear for other gauge groups.

As is nicely explained in [22], the number of normalizable ground states is given as the low temperature limit of the partition function

$$\int dx \lim_{\beta \rightarrow \infty} \text{tr} e^{-\beta H}, \quad (5)$$

but as the calculation of this quantity seems beyond reach for most systems of interest, it appears more promising to try to calculate the Witten index

$$I_w = n_B - n_F = \int dx \lim_{\beta \rightarrow \infty} \text{tr} (-1)^F e^{-\beta H}, \quad (6)$$

instead (where  $F$  is the fermion number), as this should also give the number of normalizable ground states. In case of a discrete spectrum, this partition function would be  $\beta$ -independent, so we could as well take the limit  $\beta \rightarrow 0$ ,

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<sup>1</sup>See, e.g. [26] for an overview

which is accessible in a perturbative calculation. For a continuous spectrum of the Hamilton operator, this does not work in general, since supersymmetry still pairs bosonic and fermionic modes, but the spectral density of scattering states need not be equal. In this case, the ‘boundary’ term  $I_1$  in the decomposition

$$\begin{aligned} I_w &= I_0 + I_1 \\ &= \lim_{R \rightarrow \infty} \lim_{\beta_0 \rightarrow 0} \int_{|x| < R} dx \left( \text{tr} (-1)^F e^{-\beta_0 H(x, x)} + \int_{\beta_0}^{\infty} d\beta \partial_\beta \text{tr} (-1)^F e^{-\beta H(x, x)} \right) \end{aligned} \quad (7)$$

can acquire a nonzero value. While the technique of splitting the integral into a ‘principal contribution’ from the bulk term as well as a ‘deficit contribution’ which takes the form of a boundary term [22] works remarkably well in many situations, as one frequently finds that the boundary term is zero even if it *a priori* does not seem to have to be (e.g. [23, 1, 9]), this is not the case in the systems at hand. Nevertheless, the boundary term has been calculated for  $SU(2)$  in [22], and reasons (that are based on the (heuristic) assumption that for the calculation of this deficit term the  $D0$ -branes can effectively be treated as identical freely propagating particles, as in [28]) have been given in [11] that this deficit term should be

$$I_1^{D=10}(SU(N)) = - \sum_{m|N, m>1} \frac{1}{m^2} \quad (8)$$

for the  $\mathcal{N} = 1$  models derived from  $D = 10$   $SU(N)$  gauged SYM, while one expects the value

$$I_1^{D=4}(SU(N)) = I_1^{D=6}(SU(N)) = -\frac{1}{N^2} \quad (9)$$

for the  $\mathcal{N} = 4$  and  $\mathcal{N} = 2$  models derived from  $D = 4$ , resp.  $D = 6$   $SU(N)$  gauged SYM.

Employing the mass deformation method that has been developed in [27, 21], Kac and Smilga [15] showed via group-theoretical means that under the hypothesis that no large mass bound state becomes non-normalizable as the zero-mass limit is taken, the number of normalizable ground states should be given by certain simple partition functions:

$$\begin{aligned} I_w^{D=10}(SO(N)) &= \# \text{ partitions of } N \text{ into mutually distinct odd parts} \\ I_w^{D=10}(Sp(N)) &= \# \text{ partitions of } 2N \text{ into mutually distinct even parts} \\ I_w^{D=10}(G_2) &= 2 \\ I_w^{D=10}(F_4) &= 4 \\ I_w^{D=10}(E_6) &= 3 \\ I_w^{D=10}(E_7) &= 6 \\ I_w^{D=10}(E_8) &= 11 \end{aligned} \quad (10)$$

Independent arguments that lead to the same  $SO(N)$  and  $Sp(2N)$  multiplicities have been presented in [12] which are based on an analysis of the Hilbert space of a chiral fermion that is constructed from D0 branes at an orientifold singularity.

The bulk term  $I_0$  is – at least in principle – independently accessible via a generalization [25] of the BRST deformation method which was devised by Moore, Nekrasov, and Shatashvili [19] in order to greatly simplify the calculation of the corresponding partition functions by adding terms to the action which break all but one supersymmetry (so that the partition functions do not change)<sup>2</sup>. Concerning the boundary term, the method of [11] has been generalized in [15] to other gauge groups, but it was found that the expected Witten index  $I_w = I_0 + I_1$  could not be obtained that way [25], indicating a failure of the assumption of the validity of the free Hamiltonian approach used in [11]. Nevertheless, for all the  $D = 4, 6, 10$  cases investigated previously, the expected vacuum degeneracies support the hypothesis [25] that the boundary term is (with the possible exception of  $I_1^{D=10}(G_2)$ ) always a small number in the interval  $[-1; 0]$ , and hence a prediction of the Witten index should be possible from the bulk index alone.

A first evaluation of  $I_0^{D=10}$  for special orthogonal and symplectic groups employing the Moore method has been performed in [25]; there as well as in further work on  $I_0^{D=4}$  [20], the algebraic bulkiness of bulk Witten index calculations was pointed out, and indeed, even for the  $I_0^{D=4}$  case, going to ranks far beyond 3 already required specialized term manipulation code to be written. In the following, we want to review the operational issues of the application of the Moore method to the even far more involved  $D = 10$  case and present techniques (some conservative, some speculative) that allow the calculation to be taken to regular simple groups of rank 4 and 5, and also give arguments that such a direct approach is barely feasible for simple groups of rank  $\geq 6$  employing currently available computer technology.

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<sup>2</sup>It was verified numerically in [16] for models obtained from  $D = 4$  SYM with low-rank gauge groups that this method seemingly also works for other groups besides  $SU(N)$

## 2 Computational aspects

The heat kernel calculation [28] of the bulk contribution to  $I_w$  requires evaluation of the partition function<sup>3</sup>

$$\begin{aligned}
I_0 &= \frac{1}{\mathcal{F}_G} \mathcal{Z}_{D,G}^{\mathcal{N}}, \\
\mathcal{Z}_{D,G}^{\mathcal{N}} &= \int \prod_{a=1}^{\dim G} \prod_{\mu=1}^D \frac{dX_\mu^a}{(2\pi)^{1/2}} \prod_{\gamma=1}^{\mathcal{N}} d\psi_\gamma^a \times \\
&\times \exp \left( \frac{1}{4g^2} \text{Tr} [A_\mu, A_\nu] [A_\mu, A_\nu] + \frac{1}{2g^2} \text{Tr} \psi_\alpha [\Gamma_{\alpha\beta}^\mu X_\mu, \psi_\beta] \right) \\
&= \int \prod_{a=1}^{\dim G} \prod_{\mu=1}^D \frac{dX_\mu^a}{(2\pi)^{1/2}} \text{Pf} (-i f^{bcd} \Gamma \cdot X^d) \exp \left( \frac{1}{4g^2} \text{Tr} [A_\mu, A_\nu] [A_\mu, A_\nu] \right)
\end{aligned} \tag{11}$$

where the fermionic degrees of freedom have been integrated out, yielding a homogeneous Pfaffian, cf. [17]. These partition functions  $\mathcal{Z}_{D,G}^{\mathcal{N}}$ , which come from the reduction of SYM to zero dimensions (see e.g. [2]) have been dubbed ‘Yang-Mills integrals’. The factor  $\mathcal{F}_G$  that relates it to the bulk Witten index is basically the effective gauge group volume, see [25] for details.

The BRST deformation technique of [19] greatly simplifies this to the calculation of the integral [25]

$$\begin{aligned}
I_0^{D=10}(G) &= \frac{|Z_G|}{|W_G|} \left( \frac{(E_1 + E_2)(E_2 + E_3)(E_3 + E_1)}{E_1 E_2 E_3 E_4} \right)^r \times \\
&\times \oint \prod_{k=1}^r \frac{dx_k}{2\pi i} \frac{\tilde{\Delta}_G(0, \vec{x}) \tilde{\Delta}_G(E_1 + E_2, \vec{x}) \tilde{\Delta}_G(E_2 + E_3, \vec{x}) \tilde{\Delta}_G(E_3 + E_1, \vec{x})}{\tilde{\Delta}_G(E_1, \vec{x}) \tilde{\Delta}_G(E_2, \vec{x}) \tilde{\Delta}_G(E_3, \vec{x}) \tilde{\Delta}_G(E_4, \vec{x})} \\
&\text{with} \\
\tilde{\Delta}_G(E, \vec{x}) &= \prod_{\vec{\alpha} \in \Delta_G} (\vec{x} \cdot \vec{\alpha} - E)
\end{aligned} \tag{12}$$

where  $r$  is the Lie group rank,  $|Z_G|/|W_G|$  is the quotient of the orders of the center and the Weyl group ( $1/(2^r r!)$  for  $B_r$ ,  $1/(2^{r-1} r!)$  for  $C_r$ ,  $1/(2^{r-2} r!)$  for  $D_r$ ,  $1/12$  for  $G_2$ ; these are the only cases we are concerned with here),  $E_{1,2,3,4}$  are auxiliary real quantities with  $\sum_j E_j = 0$  which the end result will not depend on (at least for sufficiently generic values of  $E_j$  where no ‘accidental’ merging of poles/zeros happens).  $\Delta_G$  is the set of roots of the Lie algebra  $G$ . Actually, as it stands these integrals do not make much sense, as we have poles on the real axis of integration. (Furthermore, they do not fall off fast enough towards complex infinity to rigorously justify closing the integration contour.) The correct (not yet fully justified, see [19]) interpretation of these integrals rather is the following algorithmic one:

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<sup>3</sup>Here and in the following, we will use the conventions of [25]

1. Successively eliminate all  $x_k$  ‘integration variables’ in the term

$$\begin{aligned} & \frac{|Z_G|}{|W_G|} \left( \frac{(E_1 + E_2)(E_2 + E_3)(E_3 + E_1)}{E_1 E_2 E_3 E_4} \right)^r \times \\ & \times \lim_{\epsilon \rightarrow 0+} \text{Res}_{x_1}^+ \dots \text{Res}_{x_r}^+ \frac{\Delta_G(0, \vec{x}) \Delta_G(\tilde{E}_1 + \tilde{E}_2, \vec{x})}{\Delta_G(\tilde{E}_1, \vec{x}) \Delta_G(\tilde{E}_2, \vec{x})} \times \\ & \times \frac{\Delta_G(\tilde{E}_2 + \tilde{E}_3, \vec{x}) \Delta_G(\tilde{E}_3 + \tilde{E}_1, \vec{x})}{\Delta_G(\tilde{E}_3, \vec{x}) \Delta_G(\tilde{E}_4, \vec{x})} \Bigg|_{\tilde{E}_j = E_j + i\epsilon^j} \end{aligned} \quad (13)$$

where the  $E_k, x_k$  are all treated as real, and  $\text{Res}_{x_j}^+$  picks up the residues with positive imaginary part only. (All the  $x_k$  except the one being integrated out are treated as real.)

2. Substitute  $E_4 = -E_1 - E_2 - E_3$
3. Evaluate at generic values of  $E_{1,2,3}$ . (Alternatively: simplify to find that the result does not depend on  $E_{1,2,3}$ .)

Due to the large number of factors in the denominator that occur in these formulae for all but the smallest simple groups, one has to resort to employing clever tricks to simplify the calculation, or massive computer aid (or both).

While the use of symbolic manipulation programs like Maple or Mathematica suggests itself for calculations like the ones at hand, and indeed interesting results have been obtained that way [25], the range to which these calculations can be carried by employing such systems in a head-on approach is quite limited, mostly due to the observation that (unlike the  $D = 4$  case) in the evaluation of  $(B, C, D)_n$  integrals, one seems to generically encounter poles of order  $(n - 1)$  that require forming  $(n - 2)$ th order derivatives of very large products, leading to an explosion of the number of terms generated. Nevertheless, it is obviously important to try to obtain these values for as many gauge groups as possible in order to check the validity of various assumptions that had to be made in the calculation of Witten indices, and perhaps even make further conjectures about analytical expressions, e.g. as in [20] for  $D = 4$ .

Even if one cannot do much about combinatoric explosion here, the question nevertheless arises, whether – considering the conceptual simplicity of the problem – one may be able to go a few steps further by trying to make use of as much of the structure of the calculation as possible in a dedicated program. While it is frequently possible to outperform general-purpose symbolic algebra packages by three orders of magnitude in term complexity with such an approach, more than one rabbit has to be pulled out of the hat in order to achieve a performance gain of one million or more, which we will see to be necessary here. As the underlying techniques are of interest for a far larger class of symbolic calculations, yet not widely known since there is (to the author’s knowledge) hardly any literature on the relevant issues, we want to briefly present some of the fundamental concepts one should be aware of when taking symbolic algebra to its limits. Incidentally, the calculation at hand is an almost perfect example to study these techniques.

Conventional symbolic algebra frequently (and especially in calculations like these) wastes most of its time generating intermediate quantities of very limited lifetime in dynamically allocated memory which has to be reclaimed soon after. In the following, we want to adhere to the convention to call this allocation of dynamic memory which will not explicitly be reclaimed *consing*<sup>4</sup>. As unnecessary consing forces frequent expensive calculations of reachability graphs of in-memory objects (in order to identify reclaimable space in the garbage collection process), it should as a rule of thumb be avoided where possible. Furthermore, one should keep in mind that for the presently dominant computer architecture the processor's memory interface is an important bottleneck, and hence it makes sense to try to find tight memory encodings for those pieces of data on which most of the calculation operates so that cache stalls are minimized.

One particularly striking feature of the calculation at hand is that naively integrating out a single variable makes conventional term manipulation programs first allocate huge numbers of almost similar terms, which are then checked (in an overwhelming number of cases in vain) for possible annihilations. This can be avoided by re-structuring the calculation in such a way that instead of generating and processing large amounts of individual mostly similar (yet different enough that cancellations become quite rare) terms, one and the same term backbone is destructively modified to consecutively represent every single new term generated and then do further processing on this backbone wherever possible. This technique is particularly useful when it comes to processing higher-order derivatives (where in addition we take care of performing the iterations over places of factors where to derive in such a way that re-occurring combinations of derivatives, as in  $(fg)'' = f''g + f'g' + fg'' + f'g' = f''g + fg'' + 2f'g'$  are only generated once).

One particular refinement for the present calculation is that one generally can hardly avoid creating new terms by substitution, except in the very last step (i.e. integrating out the last variable) where this is indeed feasible. Since the last step is also executed most frequently, this almost buys us an extra rank for free. (A slightly nontrivial subtlety for the  $D = 10$  calculation here is that we have to take care of the possibility of generating factors  $E_1 + E_2 + E_3 + E_4$  in both the numerator as well as the denominator. It may well happen that while accumulating the factors of a term, one intermediately encounters more such 'powers of zero' in the denominator than in the numerator.)

As calculations with exact numbers cause some systems to perform excessive number consing, it is wise to try to avoid the exact rational number data type for all those parts of a calculation where one can use with impunity more limited data types that have fast direct hardware support (like 32-bit integers)<sup>5</sup>. For our present calculation this means to represent the coefficients of linear functions as machine integers. (Clearly, one has to take care of proper handling of

<sup>4</sup>Named after the *cons* (pair), which is the most ubiquitous building block of hierarchical data structures in LISP

<sup>5</sup>Actually, most systems providing a dynamic GC need to use a few tag bits to discern between immediate and referenced (i.e. consed) values, so we typically only can use 30 bit signed machine integers on 32-bit hardware

some ‘balancing denominators’ that are generated by this somewhat artificial treatment that requires to take least common multiples at substitutions.) For all of the problems that are reachable with present computer hardware, all of the substitutions of such terms among themselves will not lead us outside the range of these integer machine data types.

A further property of the calculations under study is that when performed in the way described here, a large fraction of the contributions are zero, and in many cases, this can be detected somewhat easily without doing costly multiplications. Hence, it makes sense to pre-scan a term for being zero before processing it wherever applicable.

It is perhaps one of the less obvious properties of these Yang-Mills integrals that one can do much better than to substitute particular values for  $E_{1\dots 4}$  subject to the constraint  $\sum_j E_j = 0$  for the final evaluation; by bringing all factors to a proper lexicographical normal form ( $E_1$  contribution first if present, then  $E_2$  contribution, then  $E_3$  contribution, all after having substituted out  $E_4$ ), one observes to obtain also the correct final value by replacing every linear term by its leading coefficient.<sup>6</sup> While the largest calculations at present cannot be done without employing this trick, we do a cross-check using the more conventional method wherever this is possible. This is indicated in the last column of our table of results.

As a special refinement, one notes that in the final gathering of powers of linear factors, the most frequently encountered numerators and denominators are  $\pm 1$  as well as small powers of 2. Indeed, a further noticeable speedup can be achieved by treating these factors special (remembering one overall power of 2 as well as the resulting sign) in order to avoid unnecessary use of exact rational arithmetics.

As a basis for the implementation, the Objective Caml system [18] appears as very appealing, since it is highly portable to a variety of different platforms, contains an optimizing compiler that can generate compact standalone binaries, allows a very smooth and easy two-way integration of C libraries and code; all these qualities are highly desirable here especially since the problem at hand suggests itself to massive parallelization, perhaps by making use of donated computation time. (While rudimentary parallelization support is present in our implementation and has proven its usefulness, we do not implement such a large-scale scheme here, mainly because it is expected that one would need a prohibitively large number of volunteers to successfully do the next rank.) A further advantage may be that Ocaml code generally is perceived as much less alien by the uninitiated than LISP code. One further noteworthy issue here is the quality of the built-in exact rational number arithmetics, since there are all but obvious huge performance differences between various implementations<sup>7</sup>. While the implementation present in Ocaml 3.07 is slower than the one of CLISP

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<sup>6</sup>One of the more miraculous properties of these integrals is that one also observes to obtain the correct final value by replacing the sum in every linear factor by a product, i.e.  $\left(\sum_j c_j E_j\right)^p \rightarrow \left(\prod_j c_j E_j\right)^p$ ; this was discovered through an intermediate programming error.

<sup>7</sup>For example, Gambit Scheme large fraction arithmetics is roughly by a factor 2500 slower than the excellent one present in CLISP, which even well outperforms e.g. that of Maple.



by a factor of roughly 300, this is not a big problem, since our techniques to reduce the use of fraction arithmetics are powerful enough (at least when making use of the evaluation shortcut described above) to make the amount of time spent in arithmetics comparable to the time spent in other parts of the program. CLISP would perhaps hardly be a viable alternative despite its excellent arithmetics implementations, since it is only a byte-code interpreter system. As there are good reasons to make the code with which these calculations have been performed publicly available<sup>8</sup>, it has been included in the [arXiv.org](https://arxiv.org) preprint upload of this work<sup>9</sup>.

### 3 Results and discussion

The following table shows our new as well as the previously known values for the bulk contributions to the Witten index. Numerical approximations as well as the expected values from the Kac/Smilga hypothesis are also given. Calculation times refer to accumulated wall clock times on a stepping-9 Intel Pentium 4 CPU, 2.4 GHz, Hyperthreading enabled/ocaml 3.07, gcc 2.95, Linux kernel 2.4.24 student computer pool installation. (Values obtained on other systems have been re-scaled appropriately.) Boldface indicates new values.

Some features that deserve special attention here are the particularly large values for  $SO(8)$  and  $Sp(10)$  that would require  $I_1 < -1$ , the  $SO(9)$  and  $SO(11)$  values that would require  $I_1 > 0$ , and the dramatic explosion of calculation time; beyond some ‘trivial’ cases, increasing the rank by one costs a factor (as a rule of thumb) of roughly 1000 in CPU time in the interesting regime. Hence, it is probably not yet feasible to try to attack rank 6 for the  $(B, C, D)_n$  groups. As the  $A_n$  bulk Indices are just the expected ones, and as the  $(B, C, D)_n$  are of the right magnitude, with denominators being powers of two (this is not the case at all for individual summands), there is good reason to believe in both the validity of this approach and its implementation. However, the calculation of the  $C_4$  index by using explicit values for  $E_{1,2,3}$  seems to fail systematically for yet unknown reasons, despite a very careful analysis of the code. (This may be related to yet another bug in the ocaml compiler; this assumption is also fueled by the observation that part of the  $D_6$  calculation causes memory violations that should not be possible at all in pure ocaml for a very specific combination of processors and optimization flags.)

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<sup>8</sup>first, as we are employing some algorithmically nontrivial tricks, one cannot exclude the possibility of errors having slipped in, hence it is provided for the sake of reproducibility and checkability of all steps; second, there are many more group-theoretical tricks one may want to exploit here and incorporate into the existing code; third, one may want to carry the calculation further as soon as more computer power becomes accessible, or by employing massive parallelization; fourth, it may be instructive to see the detailed implementation of some of the techniques described above.

<sup>9</sup><http://www.arxiv.org/e-print/hep-th/0312262>

Group	$I_0^{D=10}$	num. approx.	expected $I_w$	approx. calc. time (sec)	cross-check
$SO(3) (B_1)$	$\frac{5}{4}$	1.2500	1	$2.6 \cdot 10^{-4}$	ok
$SO(5) (B_2)$	$\frac{81}{64}$	1.2656	1	$1.1 \cdot 10^{-2}$	ok
$SO(7) (B_3)$	$\frac{325}{256}$	1.26953	1	$7.8 \cdot 10^{-1}$	ok
$SO(9) (B_4)$	$\frac{15925}{8192}$	1.8671	2	$6.6 \cdot 10^2$	ok
$SO(11) (B_5)$	$\frac{59049}{32768}$	1.8020	2	$2.8 \cdot 10^6$	
$SO(4) (D_2)$	$\frac{25}{16}$	1.5625	1	$3.4 \cdot 10^{-3}$	ok
$SO(6) (D_3)$	$\frac{21}{16}$	1.3125	1	$1.9 \cdot 10^{-1}$	ok
$SO(8) (D_4)$	$\frac{6885}{2048}$	3.3618	2	$1.5 \cdot 10^2$	ok
$SO(10) (D_5)$	$\frac{3025}{1024}$	2.9541	2	$2.1 \cdot 10^5$	
$Sp(2) (C_1)$	$\frac{5}{4}$	1.2500	1	$3.2 \cdot 10^{-4}$	ok
$Sp(4) (C_2)$	$\frac{81}{64}$	1.2656	1	$9.6 \cdot 10^{-3}$	ok
$Sp(6) (C_3)$	$\frac{1175}{512}$	2.2949	2	$7.7 \cdot 10^{-1}$	ok
$Sp(8) (C_4)$	$\frac{42667}{16384}$	2.6042	2	$5.2 \cdot 10^2$	failed
$Sp(10) (C_5)$	$\frac{583755}{131072}$	4.4537	3	$1.9 \cdot 10^6$	
$SU(2) (A_1)$	$\frac{5}{4}$	1.2500	1	$3.4 \cdot 10^{-4}$	ok
$SU(3) (A_2)$	$\frac{10}{9}$	1.1111	1	$8.5 \cdot 10^{-3}$	ok
$SU(4) (A_3)$	$\frac{21}{16}$	1.3125	1	$3.7 \cdot 10^{-1}$	ok
$SU(5) (A_4)$	$\frac{26}{25}$	1.0400	1	$4.7 \cdot 10^1$	ok
$SU(6) (A_5)$	$\frac{25}{18}$	1.3889	1	$1.4 \cdot 10^4$	
$SU(7) (A_6)$	$\frac{50}{49}$	1.0204	1	$1.1 \cdot 10^7$	
$G_2$	$\frac{1375}{864}$	1.5914	2	$4.5 \cdot 10^{-2}$	ok

All in all, the new data clearly show that for other gauge groups than  $SU(N)$ , the issue of the number of vacuum states is not well understood, and more work has to be put into the determination of Witten indices. In particular, we do not have at present a useful theory to calculate the boundary contributions, and it is well conceivable that some of the assumptions behind (10) may be violated.

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